A Summary Result on Second Order Linear Differential Equations with Variable Coefficients using Adomian Decomposition Method and Power Series Method

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D.O.I: 10.56201/ijasmt.v8.no4.2022.pg1.4

Abstract
This paper considers two methods with series solutions to second order linear differential equations with variable coefficients. These series solutions are given by the Adomian decomposition method and the Power series method. The two methods provide approximate solutions with their techniques being different and unique. We will take some examples to illustrate the techniques of these mentioned methods and compare the results obtained. Included in the examples to be considered, is a case where none of the coefficients of the variables is zero.

Key words/phrases: *Adomian decomposition method, power series method, linear differential equations, variable coefficients, series solutions.*

INTRODUCTION

Nearly all physical phenomena undergoing change(s) can be represented using differential equations. Thus, it has been of research interest to find solutions to differential equations. A solution to a differential equation is a function which satisfies the differential equation when it is substituted into it. This solution is often seen as the integral of the differential equation. Generally, we are good at finding solutions to second order differential equations with constant coefficients and the ones with variable coefficients but assume the Cauchy Euler equation form. However, there is a need to also find solutions to those second order differential equations with variable coefficients and do not take the Cauchy Euler equation form, as they form the foundation of the analysis to the classical problems of mathematical physics. Hence, we want to solve these kind of equations by all kinds of techniques such as Adomian decomposition method, transformation method, reduction of order, power series, repeated integration and so on. We consider the differential equation

$$y'' + p(t)y' + q(t)y = 0;\ y(0) = A, y'(0) = B;\ - - - - - - - -(1)$$

where A, B are constants and $p(t), q(t)$ are given functions of t. To solve (1), Wilmer and Costa [1] combined the transformation and repeated iterated integration method while Dabwan and Hasan [5] modified the Adomian decomposition method and applied it. Though in the equation Dabwan and Hasan [5] considered, they choose their $q(t) = p'(t)$ and their equation is non homogenous. Other researchers like Nhawu et al [4] and Wazwaz [2] applied Adomian decomposition method but the systems they considered are either first order differential equations or second order with $p(t) = 0$. In this paper, we will provide solution to equation (1) using the Adomian decomposition method (ADM, see [3]) and the power series method (PSM). In choosing the problems to work on, we included a case where $p(t) \neq 0$. Also, we will compare the results obtained. The two methods to be used provide approximate solutions and their solutions are in series form; however, Nhawu et al [4] pointed out among other advantages that Adomian decomposition method is capable of greatly reducing the size of computational work while maintaining high accuracy of the numerical solution.

PRELIMINARIES

In this section, we give the procedures for the methods mentioned earlier which are to be used in this work.

The technique of ADM

Remark: Equation (1) is equivalent to

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0; y(0) = A, y'(0) = B. ------- (2)$$

Following the work of G. Adomian [3], we link equation (2) to

$$L_{tt}y + p(t)L_t y + q(t)y = 0; y(0) = A, y'(0) = B; -----(3)$$

where $L_{tt} = \frac{d^2}{dt^2}$; $L_t = \frac{d}{dt}$. Since L_{tt} is a differential operator, we have the inverse operator which is a two fold integral operator, that is $L_{tt}^{-1} = \int(\int .\ dt)dt$. On applying L_{tt}^{-1} in (3), we obtain

$$y = A + Bt - L_{tt}^{-1}\{ p(t)L_t y + q(t)y\}; ------ (4),$$

where $A + Bt$ is the constant of integration. Adomian solution consist of decomposing y into $\sum_{n=0}^{\infty} y_n$. Thus, (4) can be written as

$$\sum_{n=0}^{\infty} y_n = y_0 - L_{tt}^{-1}\{ p(t)L_t y + q(t)y\}.$$

Consequently, we can write

$$y_0 = A + Bt,$$
$$y_{n+1} = -L_{tt}^{-1}\{ p(t)L_t y_n + q(t)y_n\}, n \geq 0.$$

The technique of PSM

This method gives solution to (1) as

$$y(t) = \sum_{k=0}^{\infty} a_k t^k , ------- (5)$$

where $a_k , k \geq 0$ are coefficients to be determined. From (5), we get $y'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$ *and* $y'' = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2}$ which we substitute into (1). Next, we make the coefficients of t equal and let the starting point for the range k to be the same. Then, we get a recursive formula with which we generate each coefficient a_k.

APPLICATIONS

Here, we apply ADM and PSM to given problems separately and for comparison purpose, we give answer to each to the seventh power of t.

Problem 1: $y'' - ty = 0;\ y(0) = 1, y'(0) = 1.$

Taking the procedure above for ADM, Problem 1 is equivalent to

$$L_{tt}y - ty = 0;\ y(0) = 1, y'(0) = 1.$$

Applying L_{tt}^{-1} and decomposing y, we obtain

$$y_0 = 1 + t,$$
$$y_{n+1} = L_{tt}^{-1}\{ty_n\}, n \geq 0.$$

Thus,

$$y_1 = \frac{t^3}{3.2} + \frac{t^4}{4.3} \equiv \frac{t^3}{3!} + \frac{2t^4}{4!}$$
$$y_2 = \frac{t^6}{6.53.2} + \frac{t^7}{7.64.3} \equiv \frac{4t^6}{6!} + \frac{10t^7}{7!}.$$

Hence, $y(t) = 1 + t + \frac{t^3}{3!} + \frac{2t^4}{4!} + \frac{4t^6}{6!} + \frac{10t^7}{7!} \dots$

Taking the procedure for PSM, we obtain $2a_2 = 0 \Rightarrow a_2 = 0$ and the recursive formula

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}, where\ k = 1,2,3, \dots$$

Recall:

$$y(t) = \sum_{k=0}^{\infty} a_k t^k\ and\ y'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$$

Thus, at $y(0) = 1$, we obtain $a_0 = 1$ and at $y'(0) = 1$, we obtain $a_1 = 1$. Using the recursive formula above, we have

k	1	2	3	4	5
a_{k+2}	$\frac{1}{3.2}$	$\frac{1}{4.3}$	0	$\frac{1}{6.5.3.2}$	$\frac{1}{7.6.4.3}$

Problem 2: $y'' + 3ty' + (t+1)y = 0;\ y(0) = 1, y'(0) = 2.$

Using ADM: Problem 2 is equivalent to

$$L_{tt}y + 3tL_t y + (t+1)y = 0;\ y(0) = 1, y'(0) = 2.$$

Applying L_{tt}^{-1} and decomposing y, we obtain

$$y_0 = 1 + 2t,$$
$$y_{n+1} = -L_{tt}^{-1}\{3tL_t y_n + (t+1)y_n\}, \qquad n \geq 0.$$

Thus,

$$y_1 = -\left\{\frac{t^4}{3.2} + \frac{3t^3}{2} + \frac{t^2}{2}\right\},$$
$$y_2 = \frac{20t^7}{7!} + \frac{88t^6}{6!} + \frac{93t^5}{5!} + \frac{7t^4}{4!},$$
$$y_3 = -\left\{\frac{160t^{10}}{10!} + \frac{264t^9}{9!} + \frac{790t^8}{8!} + \frac{1523t^7}{7!} + \frac{91t^6}{6!}\right\}.$$

Hence, $y(t) = 1 + 2t - \frac{t^2}{2!} - \frac{9t^3}{3!} + \frac{3t^4}{4!} + \frac{93t^5}{5!} - \frac{3t^6}{6!} - \frac{1503t^7}{7!} + \cdots$

Using PSM, we obtain $a_2 = -\frac{a_0}{2}$ and the recursive formula

$$a_{k+2} = -\left[\frac{(3k+1)a_k + a_{k-1}}{(k+2)(k+1)}\right], where\ k = 1,2,3, \ldots$$

The values of a_k are as follow:

K	0	1	2	3	4	5	6	7
a_k	1	2	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{4.2}$	$\frac{31}{5.4.2}$	$-\frac{1}{6.5.4.2}$	$-\frac{501}{7.6.5.4.2}$

CONCLUDING REMARKS

The Adomian decomposition and power series methods have been applied to solve second order linear differential equations with variable coefficients. It is observed that the ADM produced its results by using integrals while the results for PSM were produced by the use of generated recursive formulas. From the results obtained, both results are equivalent, but the ADM results were produced with fewer iterations. Thus, it can be said that the computational work done while using ADM is lesser. However, it is observed that when $p(t) \neq 0$, the computational work done to produce the results for ADM seems more tedious. In line with the work of some other researchers, ADM is easier to implement and converges faster. Though, we recommend the method for cases were $p(t) = 0$, this is to avoid the difficulties encountered when the method was employed for the case where $p(t) \neq 0$.

Competing Interest

The authors declare that they have no competing interest.

Contribution

Both authors read and approved the final manuscript.

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